Directed rigidity and bootstrap percolation in 1+1 dimensions

Marcio Argollo de Menezes* and Cristian F. Moukarzel†
Instituto de Física, Universidade Federal Fluminense, CEP 24210-340, Niteroi, RJ, Brazil
(Received 6 May 1999)

We study directed rigidity percolation (equivalent to directed bootstrap percolation) on three different lattices: square, triangular, and augmented triangular. The first two of these display a first-order transition at $p=1$, while the augmented triangular lattice shows a continuous transition at a nontrivial p_c. On the augmented triangular lattice we find, by extensive numerical simulation, that the the directed rigidity percolation transition belongs to the same universality class as the directed percolation. The same conclusion is reached by studying its surface critical behavior, i.e., the spreading of rigidity from finite clusters close to a nonrigid wall. Near the discontinuous transition at $p=1$ on the triangular lattice, we are able to calculate the finite-size behavior of the density of rigid sites analytically. Our results are confirmed by numerical simulation.
[S1063-651X(99)19210-9]

PACS number(s): 05.70.Jk, 05.70.Ln, 64.60.Ak

I. INTRODUCTION

Central-force rigidity percolation (RP) [1–21] is the mechanical equivalent of the usual percolation problem [22–26]. In RP forces (vectors) must be transmitted instead of scalars. This problem has received increased attention recently, following the development of mean-field theories [12,16,18] as well as of powerful combinatorial algorithms [27–30] for its numerical study [13–17,19–21]. As a result of these efforts, a deeper understanding of the rigidity transition has emerged, although some open questions remain.

Bethe lattice calculations for RP [16,18] with an adjustable number g of degrees of freedom at each site have been used to obtain the behavior of the spanning cluster density $P_\infty(p)$ as a function of p, the dilution (bond or site) parameter. For $g=1$ one has usual (scalar) percolation, displaying a continuous transition with $\beta^{MF}=1$. But for any $g>1$, the order parameter P_∞ has a discontinuity at a finite critical value p_c. Thus the rigidity transition is discontinuous for $d\to\infty$ [18,19]. Other MF approximations also predict a first-order RP transition [12].

On triangular lattices, on the other hand, there is a divergent correlation length and the RP transition is *second order* [13–17,20,21], but in a different universality class than usual percolation [20,21]. Some of the numerical evidence in two dimensions (2D) is consistent with a small discontinuity in the order parameter P_∞, or a very small value for β, but the precise interpretation of this evidence is still a matter of debate [17]. In three dimensions, numerical simulations show [31] that the rigidity transition is second order, with a value of β clearly larger than zero. This is somewhat surprising since the rigidity transition is first order ($\beta=0$) in the limit of infinite dimensionality [16,18]. It is thus at present unclear in which fashion the RP transition becomes discontinuous as the dimensionality increases. Is there something like an upper critical dimension for RP, beyond which it is first order? Or does it get increasingly ''first order'' (i.e., $\beta\to 0$) as $d\to\infty$? This analysis is further complicated by the fact that the character of the transition is *lattice* dependent. Hypercubic lattices in which sites have d degrees of freedom each cannot be rigid if they are diluted, but are rigid if undiluted and if they have appropriate boundary conditions. Thus on hypercubic lattices the RP transition is ''trivially first order'' at $p_c=1$, in any dimension.

Similar considerations apply to directed lattices. Bethe lattices are directed by construction, since there is only one path between any two given sites. On directed lattices, rigid connectivity takes a particularly simple form. Imagine a rigid boundary to which a site with g degrees of freedom must be rigidly attached by means of rotatable springs (central forces). Each spring, or bond, restricts one degree of freedom. Thus the minimum number of bonds required to completely fix this site is g. Propagation of rigidity on directed lattices is then defined in the following terms: a site (with g degrees of freedom) at ''time'' t is rigidly connected to a boundary at $t=0$ if and only if it has g or more neighbors at *earlier* times who in turn are rigidly connected to the boundary. Thus, in contrast to *undirected* rigidity, which requires complex algorithms [27–30] that presently limit the maximum sizes to approximately 1.6×10^7 sites [21,20], directed rigidity percolation (DRP) can be studied by means of a simple numerical procedure, and on much larger systems.

It is interesting to notice that, on any *directed* lattice, rigidity percolation is equivalent to *bootstrap percolation* (BP), a modified percolation problem in which a site belongs to a cluster if at least m of its neighbors also do [32–40]. Bootstrap percolation on undirected lattices attempts to describe certain systems in which atoms behave magnetically only if they are surrounded by a ''large enough'' number of magnetic neighbors. A second reason for interest in BP is the search for novel critical behaviors in percolation [37,38], but the present understanding of this problem indicates that BP is either ''trivially first order'' with $p_c=1$ or second order and in the universality class of scalar percolation [39–41].

Early studies of semidirected $m=2$ BP on square lattices seemed to indicate a transition at a nontrivial p [34,36], but rigorous arguments [35] later showed that $p_c=1$ in this case. To our knowledge there are no published studies of directed

*Electronic address: marcio@if.uff.br
†Electronic address: cristian@if.uff.br

bootstrap percolation (DBP) displaying a second-order transition.

It has been recently conjectured [42] that any continuous transition in a nonequilibrium process with a scalar order parameter, and a nonfluctuating, nondegenerate absorbing state must be in the same universality class as directed percolation (DP) [42–51]. According to this, DRP-DBP would belong to the DP universality class in all dimensions for which it has a continuous transition.

It is thus interesting to study RP on finite-dimensional directed lattices of increasing dimensionality, both to test this conjecture and to understand in which fashion DRP becomes discontinuous as $d\to\infty$. In this article we report on our results for directed rigidity percolation (DRP, equivalent to DBP) on several $(1+1)$-dimensional lattices displaying first- and second-order phase transitions.

If the DRP transition is second order, we pay particular attention to the determination of critical indices associated with the spreading of rigidity, as described in the following [44,45]. As is usual in the study of directed processes, we define $D(p)$ to be the asymptotic density of ''active'' (rigid) sites, which is equivalent to the probability $P(p)$ that, at large times t, a randomly chosen point will be rigidly connected to a totally rigid boundary at $t=0$. If the dilution parameter p is lower than a critical value p_c, rigidity does not propagate and $P(p)=0$. If the transition is second order, immediately above p_c one has $P(p)\sim(p-p_c)^{\beta^{dens}}$.

If the evolution starts from a *finite* rigid cluster or ''seed'' at $t=0$ instead of a rigid boundary, one defines $P_a^{seed}(t,p)$ as the probability that the cluster grown from this seed will still be ''active'' at time t. If the transition is second order, $P_a^{seed}(t,p)\sim(p-p_c)^{\beta^{seed}}$ for $p\to p_c^+$ and $t\to\infty$. At p_c this quantity decays as

$$P_a^{seed}(t,p_c)\sim t^{-\delta}, \tag{1}$$

with $\delta=\beta^{seed}/\nu_\parallel$ and $\nu_\parallel$ the temporal (or parallel) *correlation length* exponent: $\xi_\parallel\sim|p-p_c|^{-\nu_\parallel}$.

The typical width w of a cluster grown from a finite seed at p_c behaves as

$$w(t)\sim t^{\chi}, \tag{2}$$

where $\chi=\nu_\perp/\nu_\parallel$ and $\nu_\perp$ is the critical index associated with the decay length $\xi_\perp$ of perpendicular or ''space'' correlations: $\xi_\perp\sim|p-p_c|^{-\nu_\perp}$. Averages are taken only over clusters still alive at time t. Finally, the average mass of a cluster grown from a finite seed at p_c behaves as

$$M_{seed}(t)\sim t^{\tilde{\eta}}, \tag{3}$$

where $\tilde{\eta}=(\nu_\parallel+\nu_\perp-\beta^{dens})/\nu_\parallel$ [45].

For comparison we also simulate numerically usual directed percolation (DP, which corresponds to $g=1$). In the DP case a simple argument shows that $\beta^{dens}=\beta^{seed}$ because of time-reversal symmetry: consider for simplicity bond dilution and choose an arbitrary point x at time t. Any configuration of occupied bonds connecting (x,t) to the boundary at $t=0$, and thus contributing to $P(p,t)$, when reflected in the time direction, contributes to $P_a^{seed}(t,p)$ if now a pointlike seed is located at x. Since both the original and the time-inverted configuration have the same probability, $P(p,t)=P_a^{seed}(p,t)$ exactly for bond-diluted DP, therefore $\beta^{dens}=\beta^{seed}$. Notice that this equality implies $\tilde{\eta}+\delta-\chi=1$.

Although no such time-reversal symmetry exists for DRP, we find that $\beta^{dens}=\beta^{seed}$ also in this case. Furthermore, we find that DRP belongs to the DP universality class, i.e., has exactly the same critical indices. Thus, there is no separate universality class for directed rigidity percolation as there is for *undirected* rigidity percolation. This is consistent with a recent conjecture [42] according to which any nonequilibrium process with a single absorbing state will belong to the same universality class as DP.

We also studied the surface critical behavior [45–47,50,51] of DRP, by means of simulations in the presence of an absorbing boundary. In the DP case, the presence of the absorbing wall is known to only modify the survival exponent β^{seed}. We find that this is also the case for DRP, and the new exponent is also consistent with the one obtained for DP with a wall.

In Sec. II we present our numerical results for DRP on directed lattices, with and without absorbing walls, and estimate the relevant critical indices associated with the second-order transitions. Section III describes DRP on a directed triangular lattice. This case has a first-order transition at $p=1$, and can be solved exactly for $p\to1$.

II. NUMERICAL SIMULATIONS

In order to simulate DRP we store a binary variable per site, indicating whether the given site is or is not rigidly connected to the boundary at $t=0$. We use the by now standard techniques of multispin coding (MSC) [52], which allows us to store 64 binary variables (on a DEC-Alpha processor) in an integer word, and also to update all of them simultaneously. A brief description of the algorithm for the case of a site-diluted triangular lattice now follows. Let `word[t]` be an integer word that will contain the states of 64 site variables at time t. Assume that `left`, `down`, and `right` code the rigid state of the corresponding left, down, and right neighbors at earlier times. A bit set to 1 in any of these words means that a site is rigidly connected to the boundary at $t=0$. We first construct an auxiliary word `aux` whose bits are set to 1 if the corresponding site has two or more of its neighbors at earlier times set to 1: `aux=(left.AND.right).OR.(left.AND.down).OR.(right.AND.down)`. This example corresponds to $g=2$. Site dilution is now implemented by masking this word with a random integer word `random`, whose bits are set to 1 with probability p. Therefore the rigid state of 64 sites at time t is determined as `word[t]=aux.AND.random`. Since interactions are short ranged in the time direction, we only need to keep in memory a maximum of three consecutive lines of the system and we do this by means of three linear arrays, which are reused periodically.

We have considered three different oriented lattices: square, triangular, and five neighbor (5n) lattice (see later). The first two display trivial behavior (i.e., rigid only at $p=1$) and the third presents a continuous DRP transition at a nontrivial p_c. In the first place we discuss DRP on a square lattice as depicted in Fig. 1(a). Each site at time t has two

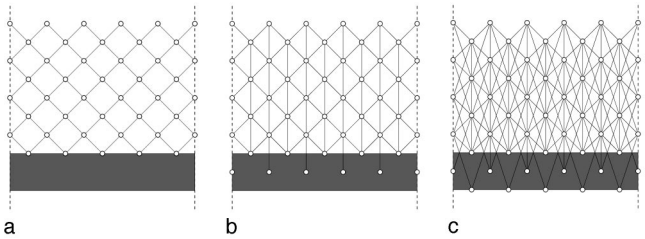

FIG. 1. Lattices used for directed rigidity percolation studies in this work: (a) square lattice, (b) triangular lattice, and (c) five-neighbor lattice. All examples are shown here undiluted ($p=1$). Rigidity propagates upwards in ''time'' from the rigid boundary (gray) at $t=0$. A site is defined to be rigidly connected to this boundary if it has two or more neighbors at *earlier* times, which in turn are connected.

neighbors at time $t-1$. This is the minimum number of neighbors needed for rigidity with $g=2$ and thus any amount of dilution is enough to impede propagation of rigidity. Therefore square lattices are not rigid at any $p<1$. If $p=1$, rigidity propagates only if boundary conditions are appropriate (e.g., rigid, or periodic, but not open). Any finite rigid cluster of size l shrinks to zero in l time steps, as shown in Fig. 2(a). The same would happen on directed d-dimensional hypercubic lattices if $g=d$.

We next consider the triangular lattice, oriented as shown in Fig. 1(b). Each site has three neighbors at earlier times. Despite the number of neighbors being larger than the minimum required (two), this lattice is also unable to propagate rigidity if diluted by any amount. To see why this is so, consider Fig. 2(b), where one starts from a finite cluster of rigid sites (black sites) at $t=0$. If the lattice is undiluted ($p=1$), this cluster would just propagate unchanged in ''time.'' If the lattice is diluted by any amount, this rigid cluster would gradually shrink and eventually disappear. Thus for this lattice, $p_c=1$, the same as for the square lattice. In contrast, finite-size effects are expected to be quite strong on the triangular lattice, since the lifetime of a finite rigid cluster diverges as $p\to 1$, no matter its original size. Also boundary effects are different since now propagation of rigidity can exist without periodic boundary conditions. We discuss this case in detail in Sec. III.

In order to have a nontrivial p_c for DRP, we use triangular lattices augmented with two further bonds per site. These extra bonds connect layers t and $t-2$, as shown in Fig. 1(c). This makes a total of five neighbors per site and we call this the $5n$ lattice for simplicity. Now consider what happens when starting from a finite cluster of rigid sites on an undiluted $5n$ lattice. As shown in Fig. 2(c), the size of the rigid cluster *expands* in time with a constant angle if $p=1$. Thus there will be a nontrivial value p_c, above which rigidity propagates forever. We find that the DRP transition is second order on this lattice. For comparison we also simulate DP on the square lattice.

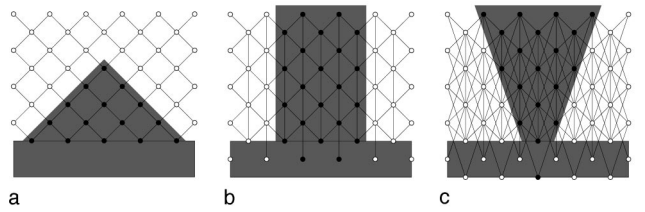

FIG. 2. Propagation of rigidity from finite clusters. The starting configuration is a finite sequence of contiguous rigid sites (black dots). (a) On the square lattice, the size of this rigid cluster shrinks in time. (b) On the triangular lattice, the rigid cluster's size remains constant in time if the lattice is undiluted, but shrinks in time for any $p<1$; thus $p_c=1$ for this case. (c) On the undiluted $5n$ lattice, rigid clusters grow in time. Therefore for this lattice there is a nontrivial value p_c (depending on the type of dilution, i.e., bond or site) above which rigidity propagates forever.

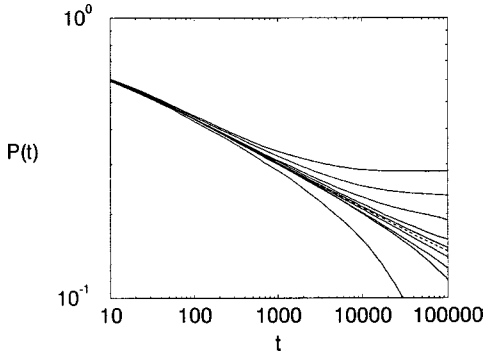

FIG. 3. Survival probability for DRP clusters grown from finite seeds on site-diluted $5n$ lattices with (from bottom to top) $p=0.704\,00$, $0.704\,80$, $0.704\,90$, $0.705\,00$, $0.705\,05$ (dashed); $0.705\,10$, $0.705\,20$, $0.705\,50$, $0.706\,00$, and $0.707\,00$. Averages were taken over 10^5 realizations on systems of width 3840 sites.

We next discuss our numerical results for DRP on the $5n$ lattice, and compare them to DP on the square lattice. We typically start our simulations from a finite seed of contiguous rigid sites and let the system evolve for 10^5 time steps (or until all activity dies out) and measure the survival probability P_a^{seed}, cluster width w, and average mass M_{seed} as a function of time.

In a first set of simulations we estimate the critical density p_c for DRP on site-diluted $5n$ lattices, by measuring $P_a^{seed}(t)$ at different values of p and identifying the one for which the asymptotic behavior is closest to a straight line in a log-log plot (Fig. 3). From these data we estimate $p_c^{DRP}=0.705\,05\pm 0.000\,05$. In contrast to P_a^{seed}, which shows appreciable curvature for off-critical values of p, the slopes of the cluster mass M_{seed} and the meandering width $w(t)$ in a similar log-log graph show little variation when $p\neq p_c$. For DP on site-diluted square lattices, we use the estimate [49] $p_c=0.644\,70$.

Figure 4(a) shows $P_{seed}(t)$ for DRP ($5n$ lattice) and DP (square lattice) at their respective critical values. Assuming power-law corrections to Eq. (1), we fit $P_{seed}(p_c,t)=at^{-\delta}(1+bt^{-\omega})$ and find $\delta^{DRP}=0.15\pm 0.01$ and $\delta^{DP}=0.16\pm 0.02$.

The cluster mass $M(t)$ and the meandering width $w(t)$ behave as shown in Figs. 4(b) and 4(c), respectively. From these data we estimate $\tilde{\eta}^{DRP}=1.47\pm 0.01$, $\tilde{\eta}^{DP}=1.47\pm 0.01$, $\chi^{DRP}=0.633\pm 0.005$, and $\chi^{DP}=0.631\pm 0.005$. These estimates are consistent with the more precise values [45,49] $\delta^{DP}=0.1594$, $\tilde{\eta}^{DP}=1.4732$, and $\chi^{DP}=0.6327$, suggesting that DRP and DP are in the same universality class.

In order to further test whether DRP has the same critical behavior as DP, we also studied DRP in the presence of an

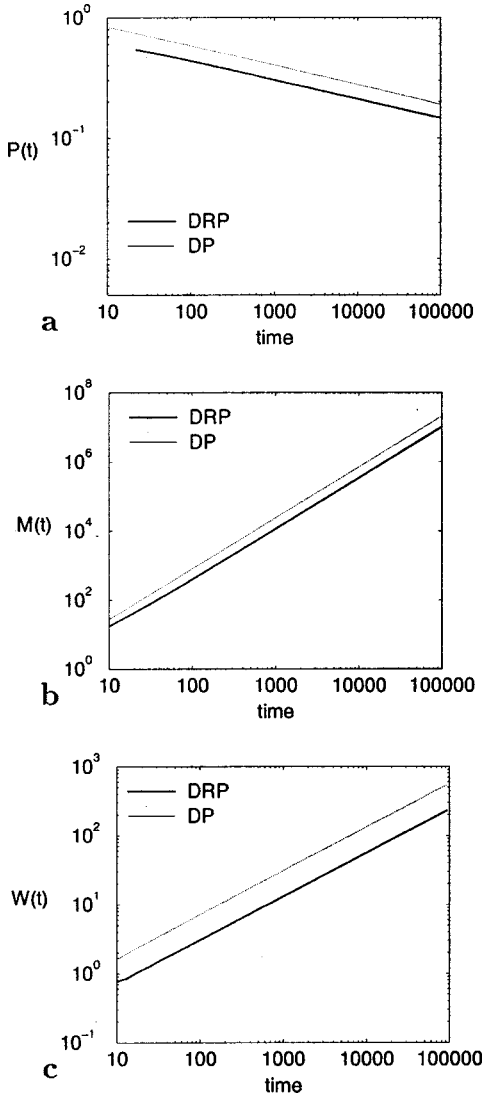

FIG. 4. Critical behavior obtained from finite seeds at the second-order transition for DRP on site-diluted $5n$ lattices and for DP on site-diluted square lattices. (a) Survival probability; (b) cluster mass; (c) meandering width.

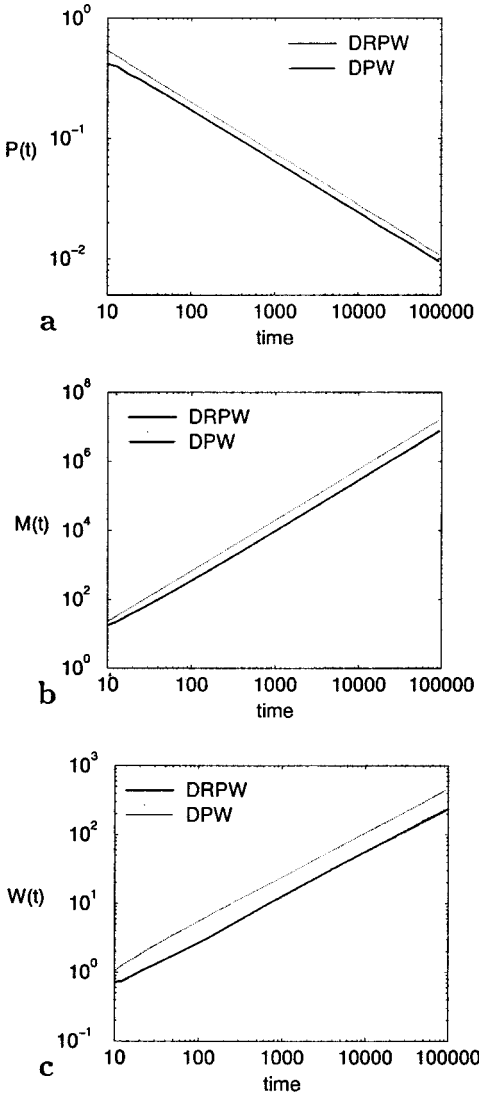

FIG. 5. Same as Fig. 4, but now finite clusters start near an absorbing wall at $x=0$. (a) Survival probability; (b) cluster mass; (c) meandering width.

absorbing wall (DRPW). For DP with an absorbing wall (DPW) it is known [45–47,50,51] that the survival exponent β^{seed} is replaced by β_1^{seed}, while $\nu_\parallel$ and $\nu_\perp$ remain unchanged. Therefore only δ is expected to change due to the presence of the absorbing wall. Our results are displayed in Fig. 5, and from them we obtain $\delta^{DRPW}=0.423\pm0.003$, $\tilde{\eta}^{DRPW}=1.48\pm0.01$, and $\chi^{DRPW}=0.62\pm0.02$. Notice that δ^{DRPW}, $\tilde{\eta}^{DRPW}$, and $-\chi^{DRPW}$ no longer add up to 1, since β^{dens} and β_1^{seed} are independent exponents. These results are entirely consistent with the values obtained for DPW by other authors [45,46].

III. DRP ON THE TRIANGULAR LATTICE

This case is marginal as already advanced, since any amount of dilution will destroy rigidity and thus $p_c=1$, but on the other hand the lifetime of finite clusters is not finite as on the square lattice, but diverges as $p\to1$. As we show now, it is possible to obtain finite-size effects analytically for $q=(1-p)\ll1$.

Assume one starts from a completely rigid boundary at $t=0$, on a triangular lattice of infinite width [Fig. 6(a)]. Let $q=(1-p)\ll1$ be the dilution parameter. We do not need to specify for the moment whether we are dealing with bond or

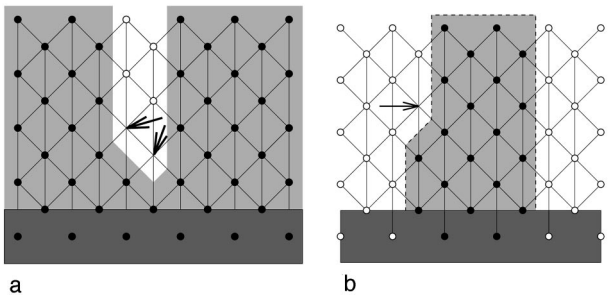

FIG. 6. The directed triangular lattice presents a first-order transition at $p=1$, which can be described in simple terms. Arrows indicate absent sites, assuming site dilution. When $p<1$ (a) defects (pairs of adjacent nonrigid sites) nucleate at rate $\rho(p)$ per unit length, giving rise to nonrigid regions that (b) widen in time with velocity $v(p)$.

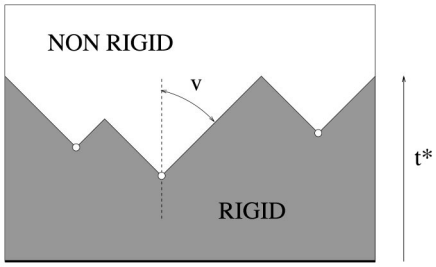

FIG. 7. Defects (empty circles) nucleate at rate $\rho(p)$ per unit length, and the resulting ''defect walls'' widen in time with velocity $v(p)$, eventually covering the entire system in a time of order t^*.

site dilution. For short times all sites are rigidly connected to the lower boundary, but soon some nonrigid sites, or ''defects,'' will appear in the presence of either bond or site dilution. The smallest possible defect is a single nonrigid site, which happens with probability q per site on site-diluted lattices (one missing site) and with probability $3q^2p+q^3 \approx 3q^2$ per site on bond-diluted lattices (two or three missing bonds). This single defect ''heals'' immediately since each site above this one has three predecessors, but needs only two rigid ones in order for itself to be rigid.

A nonhealing defect (in the following simply a defect) is created if two sites connected by a diagonal bond are simultaneously nonrigid, as in Fig. 6(a). All sites directly above these will have only one rigid neighbor and thus fail to be rigid, creating a ''defect wall.'' Assume that these paired defects are nucleated with density $\rho(p)$ per unit length (we calculate ρ later), and consider now the time evolution of the resulting defect wall.

In the absence of dilution ($q=0$), the boundaries of a nonrigid region stay unchanged in time [Fig. 6(a)]. Rigid sites directly on this boundary have only two bonds (the minimum required number since $g=2$) to rigid sites at earlier times. If one of these boundary sites fails to be rigid, all sites above it will also not be rigid. In this case the rigid boundary is displaced by one unit, as shown in Fig. 6(b). Therefore, for small but nonzero q, the rigid wall in Fig. 6(b) moves rightwards with an average velocity $v=\partial x/\partial t$, which equals the probability for a boundary site to fail to be rigid.

Neglecting fluctuations, we have a picture in which defects appear at a rate $\rho(p)$ per unit length, giving rise to nonrigid regions that widen in time with constant velocity $v(p)$. The system will become completely nonrigid when all defect regions have coalesced, as depicted in Fig. 7. This picture of the rigid-nonrigid transition is related to the polynuclear growth model [53], which has been extensively studied in the area of crystal growth. For our discussion of DRP we only need a few results, which can be derived by means of simple arguments.

Assuming one knows the cone angle $v(p)$ and the defect density $\rho(p)$, it is easy to calculate the density of rigid points $P(p,t)$ after t time steps on a system of infinite width. A point (x,t) will be rigidly connected to the rigid boundary located at $t=0$ if it has not suffered the effect of any defect; in other words, if no defect has nucleated inside a ''cone'' with downwards opening angle v and whose vertex sits at (x,t). Let $\Omega=vt^2$ be the area of this cone. Since defects nucleate randomly in space-time with density $\rho(p)$, their number inside any given area Ω is a Poisson-distributed random variable with average $\Omega\rho$. Thus

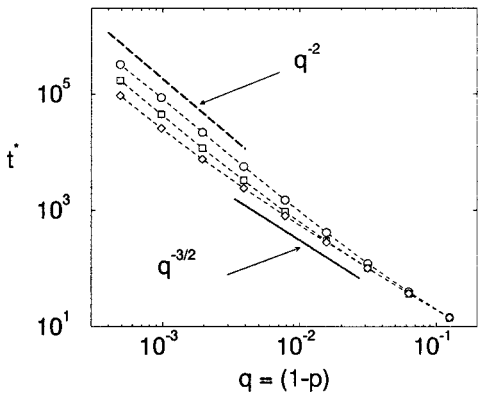

FIG. 8. Mean time for the disappearance of rigidity, on site-diluted triangular lattices of width $w=128$ (circles), 256 (squares), and 512 (diamonds), as a function of $q=1-p$.

$$P(p,t)=e^{-(t/t^*)^2}, \tag{4}$$

where

$$t^*=(v\rho)^{-1/2} \tag{5}$$

is a characteristic time for the disappearance of rigidity, on an infinitely wide system. Using similarly simple arguments it is easy to see that the mean lifetime of a *finite* rigid cluster diverges as v^{-1} as $p\to 1$ (see Fig. 8).

We now calculate $v(p)$ and $\rho(p)$, and compare the resulting prediction for t^* with our numerical results. Under site dilution, the probability per unit time for a rigid wall to be displaced by one unit is simply $v_{site}=q$. If bonds are diluted instead, one gets $v_{bond}=1-p^2\approx 2q$. In order to calculate ρ, we notice that a pair of contiguous absent sites appears with probability $\rho_{site}=2q^2$ per site and per unit time on site-diluted lattices. On bond-diluted lattices, on the other hand, creating such a pair requires at least three missing bonds. Thus $\rho_{bond}\sim q^3$. Finally one has [Eq. (5)] $t^*_{site} \sim q^{-3/2}$ and $t^*_{bond}\sim q^{-2}$. Figure 8 shows t^* as measured on site-diluted lattices. These values are obtained by integrating in time the density of rigid sites $P(p,t)$. For an intermediate range of q, it is found that $t^*\sim q^{-3/2}$, as predicted for infinitely wide systems.

On a system of a finite width w with periodic boundary conditions, a crossover to a width-dominated behavior is expected if $w\ll t^*v$; equivalently if $w\ll(v/\rho)^{1/2}$, whereupon

$$P(p,t)_{finite}=e^{-t/t^*_{finite}} \tag{6}$$

and

$$t^*_{finite}=(w\rho)^{-1}. \tag{7}$$

This regime corresponds to the defect-free area Ω becoming essentially a rectangle of height t and width w instead of a triangle of height t and base vt. According to Eq. (7), one should expect $t^*\sim q^{-2}$ for small q.

This regime is observed for $w=128$, but is less clear for larger values of w. Observation of this crossover for wider

systems is numerically difficult, since it requires one to simulate very small q values, which makes the mean rigid times too large.

IV. CONCLUSIONS

We considered directed rigidity percolation (DRP) with two degrees of freedom per site on three different $(1+1)$-dimensional lattices. This problem is equivalent to directed bootstrap percolation (DBP) with $m=2$. On the square lattice, the system is only rigid at long times if $p=1$. On triangular lattices a similar situation happens, but this case has a nontrivial behavior for $p\to 1$, which we calculate analytically and confirm by numerical simulation. The mean lifetime of rigidity on infinitely wide systems is found to diverge when $p\to 1$ as $(1-p)^{-3/2}$ for site dilution, and as $(1-p)^{-2}$ for bond dilution. The mean lifetime of a finite rigid cluster diverges on the other hand as $(1-p)^{-1}$ in both cases.

By augmenting the triangular lattice with two further bonds we define the $5n$ lattice, which has a continuous transition at $p_c^{DRP}=0.705\,05\pm 0.000\,05$ for site dilution. We measure the critical indices associated with the spreading of rigidity and find that the DRP transition belongs to the directed percolation (DP) universality class, as a recent conjecture would indicate. A similar numerical study of DRP with an absorbing wall gives exponents equally consistent with those of DP. Thus, while (undirected) rigidity percolation does not belong to the same universality class as usual percolation, the introduction of directedness makes these two problems essentially equivalent at their respective critical point, i.e., on large scales.

On $(d+1)$-directed lattices, DP is always second order (with mean-field exponents above its upper critical dimension $d_c=5$). On the other hand, Bethe lattice calculations indicate that the DRP transition is first order for large d. The only difference between these two problems is the number g of degrees of freedom per site. While DP corresponds to $g=1$ (and has a continuous transition in all dimensions), DRP corresponds to any $g>1$ and has a discontinuous transition on the Bethe lattice (infinite dimensionality). Therefore it is clear that DRP and DP, which as shown here have the same critical behavior in $d=1$, will cease to be equivalent when the space dimensionality d is increased. We are presently extending this study to larger values of d.

ACKNOWLEDGMENTS

One of us (C.M.) wishes to thank K. Lauritsen and P. Grassberger for useful discussions on DP. C.M. was supported by FAPERJ, and M.A.M by CAPES, Brazil.

[1] S. Feng and P. Sen, Phys. Rev. Lett. **52**, 216 (1984).
[2] Y. Kantor and I. Webman, Phys. Rev. Lett. **52**, 1891 (1984).
[3] M. F. Thorpe, in *Physics of Disordered Materials*, edited by D. Adler, H. Fritzsche, and S. Ovishinsky, Institute for Amorphous Studies Series (Plenum Press, New York, 1985).
[4] D. J. Bergman, Phys. Rev. B **33**, 2013 (1985).
[5] A. Day, R. Tremblay, and A.-M. Tremblay, Phys. Rev. Lett. **56**, 2501 (1986).
[6] M. F. Thorpe and E. J.Garboczi, Phys. Rev. B **35**, 8579 (1987).
[7] S. Roux and A. Hansen, Europhys. Lett. **6**, 301 (1988).
[8] S. Arbabi and M. Sahimi, J. Phys. A **21**, L863 (1988).
[9] A. Hansen and S. Roux, Phys. Rev. B **40**, 749 (1989).
[10] M. Knackstedt and M. Sahimi, J. Stat. Phys. **69**, 887 (1992).
[11] S. Arbabi and M. Sahimi, Phys. Rev. B **47**, 695 (1993).
[12] S. P. Obukov, Phys. Rev. Lett. **74**, 4472 (1995).
[13] C. Moukarzel and P. M. Duxbury, Phys. Rev. Lett. **75**, 4055 (1995).
[14] D. J. Jacobs and M. F. Thorpe, Phys. Rev. Lett. **75**, 4051 (1995).
[15] D. J. Jacobs and M. F. Thorpe, Phys. Rev. E **53**, 3682 (1996).
[16] C. Moukarzel, P. M. Duxbury and P. L. Leath, Phys. Rev. Lett. **78**, 1480 (1997).
[17] D. J. Jacobs and M. F. Thorpe, Phys. Rev. Lett. **80**, 5451 (1998); P. M. Duxbury, C. Moukarzel, and P. L. Leath, *ibid.* **80**, 5452 (1998).
[18] C. Moukarzel, P. M. Duxbury, and P. L. Leath, Phys. Rev. E **55**, 5800 (1997).
[19] P. M. Duxbury, D. J. Jacobs, M. F. Thorpe, and C. Moukarzel, Phys. Rev. E **59**, 2084 (1999).
[20] C. Moukarzel and P. Duxbury, in *Rigidity Theory and Applications*, edited by M. Thorpe and P. Duxbury (Plenum Press, New York, 1998); see e-print cond-mat/9807004.
[21] C. Moukarzel and P. M. Duxbury, Phys. Rev. E **59**, 2614 (1999).
[22] D. Stauffer and A. Aharony, *Introduction to Percolation Theory*, 2nd ed. (Taylor & Francis, London, 1994).
[23] M. Sahimi, *Applications of Percolation Theory* (Taylor & Francis, London, 1994).
[24] *Fractals and Disordered Systems*, edited by A. Bunde and S. Havlin, 2nd ed. (Springer, Berlin, 1996).
[25] J. W. Essam, Rep. Prog. Phys. **43**, 53 (1980).
[26] M. Sahimi, in *Annual Reviews of Computational Physics II*, edited by D. Stauffer (World Scientific, Singapore, 1995).
[27] Bruce Hendrickson, SIAM J. Comput. **21**, 65 (1992).
[28] C. Moukarzel, J. Phys. A **29**, 8097 (1996).
[29] D. Jacobs and B. Hendrickson, SIAM J. Comput. **137**, 346 (1997).
[30] D. J. Jacobs, J. Phys. A **31**, 6653 (1998).
[31] C. Moukarzel (unpublished).
[32] J. Chalupa, P. L. Leath, and G. R. Reich, J. Phys. C **12**, L31 (1979).
[33] M. Aizenmann and J. L. Lebowitz, J. Phys. A **21**, 3801 (1988).
[34] J. A. M. S. Duarte, Physica A **157**, 1075 (1989).
[35] R. H. Schonmann, J. Stat. Phys. **58**, 1239 (1990).
[36] A. C. D. van Enter, J. Adler, and J. A. M. S. Duarte, J. Stat. Phys. **60**, 323 (1990).
[37] J. Adler and D. Stauffer, J. Phys. A **23**, 1119 (1990).
[38] J. Adler, Physica A **171**, 453 (1991).
[39] C. M. Chaves and B. Koiller, Physica A **218**, 271 (1995).
[40] M. C. Medeiros and C. M. Chaves, Physica A **234**, 604 (1997).
[41] N. S. Branco and C. J. Silva, e-print cond-mat/9904239.
[42] H. K. Janssen, Z. Phys. **42**, 151 (1981); P. Grassberger, *ibid.*

47, 365 (1982); J. Stat. Phys. **79**, 13 (1995).
[43] W. Kinzel, in *Percolation Structures and Processes*, Annals of the Israel Physical Socity Vol. 5, edited by G. Deutscher, R. Zallen, and J. Adler (Hilger, Bristol, 1983), p. 425.
[44] P. Grassberger, J. Phys. A **22**, 3673 (1989).
[45] K. B. Lauritsen, K. Sneppen, M. Markošová, and M. H. Jensen, Physica A **247**, 1 (1997).
[46] P. Fröjdh, M. Howard, and K. B. Lauritsen, J. Phys. A **31**, 2311 (1998).
[47] K. B. Lauritsen, P. Fröjdh, and M. Howard, Phys. Rev. Lett. **81**, 2104 (1998).
[48] E. Domany and W. Kinzel, Phys. Rev. Lett. **47**, 5 (1981); **53**, 311 (1984); F. Y. Wu and H. E. Stanley, *ibid.* **48**, 775 (1982).
[49] R. J. Baxter and A. J. Guttmann, J. Phys. A **21**, 3193 (1988).
[50] J. W. Essam and D. Tanlakishani, J. Phys. A **27**, 3743 (1994).
[51] J. W. Essam, A. J. Guttmann, I. Jensen, and D. TanlaKishani, J. Phys. A **29**, 1619 (1996).
[52] Paulo Murilo Castro de Oliveira, *Computing Boolean Statistical Models* (World Scientific, Singapore, 1991).
[53] F. C. Frank, J. Cryst. Growth **22**, 233 (1974); G. H. Gilmer, *ibid.* **49**, 465 (1980); C. H. Bennet, M. Büttiker, R. Landauer, and H. Thomas, J. Stat. Phys. **24**, 419 (1981); N. Goldenfeld, J. Phys. A **17**, 2807 (1984); W. van Saarlos and G. H. Gilmer, Phys. Rev. B **33**, 4927 (1986).